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Transfinite Interpolation of Conic Triangles Using Quadric Patches

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Abstract—The use of quadric surface patches to form a continuous approximating surface to given curved surfaces was discussed in a paper by Baart and McLeod. A method was proposed for the construction of continuous patches on a covering mesh of conic segments, where each patch consists of three quadric subpatches, and each conic boundary segment interpolates three points on the given surface. This construction is not unique, and in this paper we investigate the effect of various combinations of the parameters on the resulting quadric patches. We also propose a method that will ensure that, were the given sculptured patch itself quadric, the approximation will be exact.

Keywords—Transfinite interpolation, Quadric patches, Conic triangles.

INTRODUCTION

Given a conic triangle with vertices on a curved surface, Baart and McLeod [1] developed a method for constructing three continuous quadric subpatches. Each subpatch contains one of the boundary conics as well as an additional selected interior point on the curved surface.

Two or more conics are said to be quadric-connected if there exists at least one quadric surface that contains the conics. Any two of three distinct but arbitrary conics on a nondegenerate quadric surface must have exactly two points in common [1]. These points lie on the intersection line of the planes of the relevant conics. Since each pair of conics of a conic triangle has a vertex in common, the other intersections must also be real.

To simplify the construction, the following projective transformation is used.

- The intersection of the three planes of the given conics is assumed to be a unique point, which is mapped to the origin.
- The three vertices of the conic triangle are mapped to the points $A(1,0,0)$, $B(0,1,0)$, and $C(0,0,1)$, respectively.
- The additional point on the curved surface is mapped to the point $S(1,1,1)$.

Denote the images of the three given conics by C_1, C_2, and C_3, respectively (Figure 1). Let Q_i denote the quadric subpatch through C_i for $i = 1,2,3$. Each Q_i has an algebraic representation

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$$a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5yz + a_6xz + a_7x + a_8y + a_9z + a_{10} = 0, \tag{1}$$

and has nine degrees of freedom. By imposing the condition that Q_i has to pass through C_i and the point S, we use six of the nine degrees of freedom. Therefore, each quadric subpatch has three degrees of freedom left.

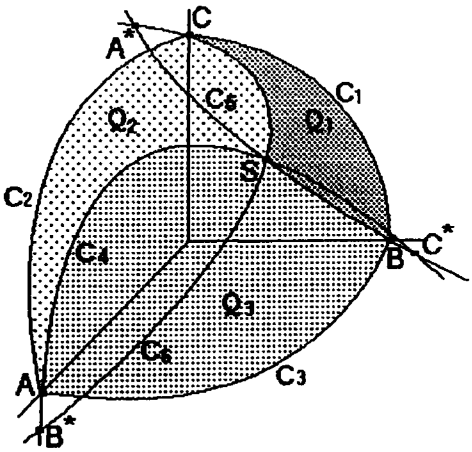

Figure 1. Quadric subpatch construction.

Let C_4, C_5, and C_6 denote the boundary conics of the three quadric subpatches as in Figure 1, and let π_4, π_5, and π_6, respectively, denote the planes of these three conics. In order for C_1, C_5, and C_6 to lie on the quadric Q_1, these three conics have to be quadric-connected. Therefore, C_1 has to intersect C_5 in a point different from B, say $C_1 \cap \pi_5$, C_1 has to intersect C_6 in a point different from C, say $C_1 \cap \pi_6$, and C_5 has to intersect C_6 in a point different from S, say $A^* = C_5 \cap C_6$. Using the same arguments for Q_2 and Q_3, the following conditions are imposed on the three interior boundary conics.

- The points A, S, $C_2 \cap \pi_4$, $C_3 \cap \pi_4$, $B^* = C_4 \cap C_6$, and $C^* = C_4 \cap C_5$ must be on C_4.
- The points B, S, $C_3 \cap \pi_5$, $C_1 \cap \pi_5$, $A^* = C_5 \cap C_6$, and $C^* = C_4 \cap C_5$ must be on C_5.
- The points C, S, $C_1 \cap \pi_6$, $C_2 \cap \pi_6$, $B^* = C_4 \cap C_6$, and $A^* = C_5 \cap C_6$ must be on C_6.

These constraints cannot, in general, be satisfied, as six conditions are imposed on each of the constructed conics. If a point $S^* = A^* = B^* = C^*$ is selected, subject to certain restrictions on its position, as discussed in [1], then each subpatch must contain the point S^*. This point will then uniquely determine the conics C_4, C_5, and C_6.

In order to ensure continuity between adjacent subpatches, a system of six linear and three nonlinear equations in nine unknowns has to be solved. The nine unknowns represent the three degrees of freedom that are left for each subpatch. By selecting the point S^*, and imposing the conditions that $S^* \subset Q_i$ for $i = 1, 2, 3$, the three nonlinear equations can be replaced by three linear equations, and the set of nine linear equations can, in general, be solved to obtain the equations of the quadric subpatches.

This construction is not unique, as the restrictions on the position of the point S^* are not very severe. In this paper, we investigate a method for selecting the point S^* in such a way that, if C_1, C_2, and C_3 are quadric-connected, with S on the quadric, then the constructed quadric patches will form part of the original quadric.

REFORMULATION OF THE PROBLEM

By selecting a point S^*, we assume that the planes π_4, π_5, and π_6 intersect in a straight line, as the points S and S^* lie on all three planes. Instead of selecting a point, we can select the direction of the line of intersection of these three planes, and then choose a point on this line.

Let the line l of intersection of π_4, π_5, and π_6 be represented by the vector

$$\mathbf{l}(t) = (1 - at, 1 - bt, 1 - ct), \tag{2}$$

where a selected value of the parameter t determines the position of S^* on this line. The line l is uniquely determined by the direction vector (a, b, c), i.e., it has two degrees of freedom, since the length of the vector is arbitrary. The selection of this vector has to be done with caution, as there are directions that lead to either an infinite number of solutions, or make it impossible to obtain continuity between adjacent patches, for general boundary conics. These directions are identified in the first three theorems.

THEOREM 1. *If $a - b - c = 0$ or $a - b + c = 0$ or $a + b - c = 0$, there will be either an infinite number of solutions, or no solution that satisfies the continuity conditions.*

PROOF. We prove the theorem for the case $a - b + c = 0$. The other two cases are similar. Assume that $a - b + c = 0$. Then the line l is given by

$$\mathbf{l}(t) = (1 - at, 1 - (a + c)t, 1 - ct). \tag{3}$$

The plane π through the points $S(1,1,1)$, $A(1,0,0)$, and $C(0,0,1)$ is given by the equation $x - y + z = 1$. By substituting $x = 1 - at$, $y = 1 - (a + c)t$, and $z = 1 - ct$, we see that the line l lies in this plane. Assume that l does not go through A or C (see Theorem 2). Then the conic C_6 contains the noncollinear points S, C, and a point S^* in this plane, and thus has to lie in this plane. Similarly, C_4 passes through the points S, A, and the point S^* in this plane, so that $\pi = \pi_4 = \pi_6$. If $C_4 \neq C_6$, the quadric Q_2 has to contain the plane through S, A, and C, and is therefore degenerate. For continuity C_2 must be the straight line AC and this will not, in general, be true. Thus, if the given conic C_2 is not a straight line, the algorithm will not yield a solution. If $C_4 = C_6$, the points A, C, S, $C_1 \cap \pi$ and $C_3 \cap \pi$ determine this conic, S^* must be chosen on this conic to make the construction possible, and Q_2 will not be unique. ∎

THEOREM 2. *If the direction vector intersects the X-, Y-, or Z-axis, there will, in general, be no solution that satisfies the continuity conditions, unless the vector passes through one of the vertices A, B, or C, in which case there are infinitely many solutions.*

PROOF. We prove the theorem for the case where the line l intersects the Z-axis. The other two cases are similar.

Assume that l intersects the Z-axis in a point different from C. Then the plane π_6 contains the Z-axis, which is therefore the intersection line of the planes of the quadric-connected conics C_2 and C_6, as well as the intersection line of the planes of the quadric-connected conics C_1 and C_6. This means that C_1 and C_2 must have two common intersections on the Z-axis (C and another point). This imposes a condition on the given conics C_1 and C_2, which will not, in general, be satisfied.

If l passes through the vertex C, the points C, S, and S^* are collinear, which means that the plane π_6 is not unique, and can rotate around the line l, which leads to infinitely many solutions. ∎

THEOREM 3. *If the direction vector passes through one of the given boundary conics there will, in general, be no solution that satisfies the continuity conditions, unless S^* is chosen on the conic, in which case there are infinitely many solutions.*

PROOF. Consider the case where the direction vector passes through a nonvertex point of the conic C_1 and where S^* does not lie on C_1. Then the quadric Q_1 contains three points on the line l, namely S, S^*, and the point on C_1. Therefore Q_1 has to contain the line l, which means that the plane π_6, which is the plane SS^*C, intersects Q_1 in the line l and a line through the point C, say m. The quadric Q_1 has only one degree of freedom left, namely the position of the point S^*

on the line l. We replace this condition with the condition that Q_1 contains the line l. This means that Q_1 is determined. In order to obtain continuity between Q_1 and Q_2—they intersect in the conic C_6, which is the product of the lines l and m, on the plane π_6—the quadric Q_2 has to contain both lines m and l. Since C_6 is fixed and C_2 and C_6 must be quadric-connected, this imposes a condition on C_2 which will not, in general, be satisfied.

If S^* is chosen as the intersection point of l and C_1, one less restriction is placed on Q_1, because we do not have to impose an extra condition to ensure that S^* is on Q_1. There are, therefore, eight linear equations in nine unknowns which will, in general, result in infinitely many solutions.

The proofs for the other two cases are similar. ∎

SELECTION OF THE PARAMETER t

In this section, a method is developed to select the position of the point S^* on the line l. We assume that a direction vector has been chosen that satisfies the restrictions implied by Theorems 1–3. We do not have enough freedom to ensure smoothness at the vertices A, B, and C, but it may be possible to select a value for t that will ensure smoothness at the point S.

We will refer to the direction vectors that satisfy the conditions in Theorems 1–3 as problematic direction vectors. It will be shown that for any nonproblematic direction vector, there is exactly one value of t that will ensure smoothness at S, unless all three the interior boundary conics (C_4, C_5, and C_6) are degenerate. It should also be noted that, in addition to avoiding problematic directions, S^* cannot be on one of the planes containing C_1, C_2, or C_3. If S^* is on one of these conics, the solution will not be unique (Theorem 3), and if not, the relevant quadric will contain this plane, which will make it degenerate, and result in discontinuity.

The following lemma is required for the proof of Theorem 4.

LEMMA 1. *Let C be a conic not containing the Z-axis in the plane $x = ky$ and passing through the origin and a variable point $T^*(0,0,t)$ on the Z-axis, and through three given noncollinear points not on the Z-axis. If the tangent vector to C at the origin is independent of the value of the parameter t, C must consist of a line through the origin and a line through T^*.*

PROOF. We can assume without loss of generality that $k = 0$. Let the three given points be $(0, y_1, z_1)$, $(0, y_2, z_2)$, and $(0, y_3, z_3)$, with $y_i \neq 0$ for $i = 1, 2, 3$, and let C be given by the equation

$$a_1y^2 + a_2z^2 + a_3yz + a_4y + a_5z + a_6 = 0, \tag{4}$$

where the coefficients a_i may depend on the parameter t. Substituting the points $(0,0,0)$ and $(0,0,t)$ into the equation, we obtain $a_6 = 0$ and $a_5 = -a_2t$, and the equation of C reduces to

$$a_1y^2 + a_2(z^2 - tz) + a_3yz + a_4y = 0. \tag{5}$$

If $a_2 = 0$, C must contain the Z-axis. We can therefore choose $a_2 = 1$. The normal direction vector at any point on C will then be given by $(0, 2a_1y + a_3z + a_4, 2z + a_3y - t)$. At the origin the normal vector will be $\mathbf{n} = (0, a_4, -t)$. Substituting the other three points into the equation for C, and solving for a_1, a_3, and a_4, we find that

$$a_4 = \frac{\alpha t + (y_1z_3 - y_3z_1)(y_1z_2 - y_2z_1)(y_3z_2 - y_2z_3)}{y_1y_2y_3(y_2z_3 - y_2z_1 - y_1z_3 - y_3z_2 + y_3z_1 + y_1z_2)}, \tag{6}$$

where α is a constant determined by the three given points. The denominator of a_4 will not be zero, since

$$y_2z_3 - y_2z_1 - y_1z_3 - y_3z_2 + y_3z_1 + y_1z_2 = 0 \tag{7}$$

implies that the three fixed points are collinear, while $y_1 = 0$, $y_2 = 0$, or $y_3 = 0$ implies that one of the fixed points lies on the Z-axis.

For the normal vector $\mathbf{n}$—and therefore, the tangent to C at the origin—to be independent of t, a_4 must contain a factor t. This means that $y_1z_3 - y_3z_1 = 0$, $y_1z_2 - y_2z_1 = 0$, or $y_3z_2 - y_2z_3 = 0$. But $y_1z_3 - y_3z_1 = 0$ implies that $(0, y_1, z_1)$, $(0, y_3, z_3)$, and $(0,0,0)$ are collinear, which means that C contains a line through the origin. Similarly, $y_1z_2 - y_2z_1 = 0$ implies that $(0, y_1, z_1)$, $(0, y_2, z_2)$, and $(0,0,0)$ are collinear, while $y_3z_2 - y_2z_3 = 0$ implies that $(0, y_2, z_2)$, $(0, y_3, z_3)$, and $(0,0,0)$ are collinear. ∎

THEOREM 4. *Let π_1, π_2, and π_3 be three distinct planes that intersect in a straight line L. Let K_1, K_2, and K_3 be three conics, not containing the line L and with at least one of the conics nondegenerate, on π_1, π_2, and π_3, respectively, with all three conics intersecting in two distinct points on L. One of these points, say T, is fixed, while the other, say T^*, can vary. Assume also that each conic has to interpolate, in addition to T, three other given points that do not lie on L, and that these three points and T, or the three points and T^*, are not collinear. Then there is a unique point T^* that will ensure that K_1, K_2, and K_3 have a common tangent plane at the point T.*

PROOF. We assume, without loss of generality, that L is the Z-axis, that T is the origin, and that π_1 is the YZ-plane. Let T^* be the point $(0,0,t)$. All three conics lie in planes with equations of the form $x = k_iy$, with $k_1 = 0$ and k_2 and k_3 nonzero constants. We can then define, for $i = 1,2,3$, the three conics as follows:

$$K_i : \{R_i = 0\} \cap \{x = k_iy\}, \tag{8}$$

where

$$R_i = a_{i1}x^2 + a_{i2}y^2 + a_{i3}z^2 + a_{i4}xy + a_{i5}xz + a_{i6}yz + a_{i7}x + a_{i8}y + a_{i9}z + a_{i10}. \tag{9}$$

For a specific point T^*, all three conics are fixed. If we ignore the planes π_1, π_2, and π_3, and consider the quadrics R_1, R_2, and R_3, we can select any four points that do not lie in the plane π_i, to be on R_i, without changing the conic K_i. Therefore, to determine a suitable R_i, we choose the points T, T^*, three given points in the plane π_i, and any four points not in this plane. We choose two of these four points on each of the planes π_j, with $j \neq i$. We therefore use, in addition to T and T^*, the following points to determine each quadric:

$$\begin{aligned}
&R_1 : (0,u_1,v_1), (0,p_1,q_1), (0,f_1,g_1), (k_2u_2,u_2,0), (k_2p_2,p_2,q_2), (k_3u_3,u_3,0), (k_3p_3,p_3,q_3),\\
&R_2 : (k_2u_2,u_2,v_2), (k_2p_2,p_2,q_2), (k_2f_2,f_2,g_2), (0,u_1,0), (0,f_1,g_1), (k_3u_3,u_3,0), (k_3f_3,f_3,g_3),\\
&R_3 : (k_3u_3,u_3,v_3), (k_3p_3,p_3,q_3), (k_3f_3,f_3,g_3), (0,p_1,0), (0,f_1,g_1), (k_2p_2,p_2,0), (k_2f_2,f_2,g_2).
\end{aligned}$$

Note that the particular point selection simplifies the algebra, but does not effect the proof in any way.

Substituting the coordinates of T, we obtain $a_{i10} = 0$.

Substituting the coordinates of T^* into R_1, we obtain $a_{19} = -ta_{13}$, and $R_1 = 0$ reduces to

$$a_{11}x^2 + a_{12}y^2 + a_{13}\left(z^2 - tz\right) + a_{14}xy + a_{15}xz + a_{16}yz + a_{17}x + a_{18}y = 0. \tag{10}$$

If $a_{13} = 0$, K_1 is the product of the Z-axis and some other line. This contradicts the assumption that K_1 does not contain the Z-axis. Therefore, we can choose $a_{13} = 1$. Similarly, we can choose $a_{23} = a_{33} = 1$.

By substituting the remaining relevant points into R_1, and solving for the a_{ij} using a symbolic mathematical package [5], we find that the denominator of the expression for R_1 factorizes into the following factors: u_1, p_1, p_2, p_3, k_2, k_3, f_1, $k_2 - k_3$, $g_1p_1 - f_1q_1 - g_1u_1 + u_1q_1 + f_1v_1 - p_1v_1$, and $u_2q_2u_3 - u_2q_3u_3 + q_3p_2u_3 - p_3q_2u_2$. If $p_i = 0$, for one or more values of i, $u_1 = 0$ or $f_1 = 0$, we have a third point on the Z-axis, which is excluded in our theorem, as that would mean that one of the given conics contains the Z-axis. If $k_2 = 0$ or $k_3 = 0$ or $k_2 = k_3$, two of the planes are the same, which is also excluded in our theorem. If

$$g_1p_1 - f_1q_1 - g_1u_1 + u_1q_1 + f_1v_1 - p_1v_1 = 0, \tag{11}$$

the points $(0, u_1, v_1)$, $(0, p_1, q_1)$, and $(0, f_1, g_1)$ lie on a straight line, which means that K_1 has to contain this line. Since this line does not contain T or T^* (assumption), K_1 must contain the Z-axis, which is not allowed.

If

$$u_2q_2u_3 - u_2q_3u_3 + q_3p_2u_3 - p_3q_2u_2 = 0, \tag{12}$$

the points $(k_2u_2, u_2, 0)$, (k_2p_2, p_2, q_2), $(k_3u_3, u_3, 0)$, and (k_3p_3, p_3, q_3) lie in the same plane. As these points were chosen arbitrarily, this situation can be avoided. Thus, the denominator can be assumed to be nonzero, and can be ignored. Similarly, the denominators of R_2 and R_3 can be ignored.

The tangent vectors to K_1, K_2, and K_3 at T can now be calculated and are, respectively, $\mathbf{r}_1 = (0, a_{19}, -a_{18})$, $\mathbf{r}_2 = (k_2a_{29}, a_{29}, -k_2a_{27} - a_{28})$, and $\mathbf{r}_3 = (k_3a_{39}, a_{39}, -k_3a_{37} - a_{38})$. After some more algebraic manipulation, we find that the coefficients a_{ij} of R_1, R_2, and R_3 are all of the form $\alpha t + \beta$, but that $\beta = 0$ for a_{i9}, with $i = 1, 2, 3$. For K_1, K_2, and K_3 to have a common tangent plane at T, the triple product of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ has to be zero [3]. This product is given by:

$$\mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3) = \begin{vmatrix} 0 & a_{19} & -a_{18} \\ k_2a_{29} & a_{29} & -k_2a_{27} - a_{28} \\ k_3a_{39} & a_{39} & -k_3a_{37} - a_{38} \end{vmatrix}. \tag{13}$$

The first and second columns both have a factor t, which means that we can take out a factor of t^2. This factor can be eliminated, as $t = 0$ will imply that $T = T^*$. This results in a linear expression in t, which will, in general, have a unique solution, unless the third column of the determinant also has a factor t. If this is the case, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ do not depend on the value of t, as a factor t can be eliminated from these vectors. According to Lemma 1, this can only happen if all three conics are degenerate, which is excluded, as we assumed that at least one of the conics is nondegenerate. ∎

We can now prove the final theorem required for the implementation of our algorithm.

THEOREM 5. *There is a unique value of t that will ensure smoothness at the point S for any given nonproblematic direction vector, provided that the interior boundary conics C_4, C_5, and C_6 are not all degenerate.*

PROOF. We show that, if we select a nonproblematic direction vector, all the conditions of Theorem 4 are satisfied. The line L is the line SS^*, the three planes are π_4, π_5, and π_6, and the three conics are C_4, C_5, and C_6.

The fact that the three planes are distinct, follows from Theorem 1. If, for example, $\pi_6 = \pi_5$, both planes will have to be the plane BCS. But then the line of intersection of the plane π_4 with these planes will lie in the plane BCS. That can only happen if the direction vector of this line was chosen as in Theorem 1.

The fact that the three conics do not contain the line L is ensured by Theorems 2 and 3. If C_6 contains the line SS^*, this line is on the quadrics Q_1 and Q_2. If this line intersects the YZ-plane, it must contain a point of C_1, which means that the direction vector was chosen through C_1 as in Theorem 3. Similarly, if the line intersects the XZ-plane, it must contain a point of C_2, and the direction is problematic (Theorem 3). If the line intersects neither of these planes, it is parallel to the Z-axis, and π_6 contains this axis. As in Theorem 2, this leads to conditions on the boundary conics C_1 and C_2 which will not, in general, be satisfied. The other two cases are similar.

The assumption that the three given points should not be collinear with S or S^* is also satisfied. The three fixed points on C_6 are C, $\pi_6 \cap C_1$, and $\pi_6 \cap C_2$. The line through C and $\pi_6 \cap C_1$ lies in the YZ-plane, while the line through C and $\pi_6 \cap C_2$ lies in the XZ-plane. For the points C, $\pi_6 \cap C_1$, and $\pi_6 \cap C_2$ to lie on a straight line, all three points must be on the Z-axis. As S is not a point on the Z-axis, and S^* cannot be chosen to be on the Z-axis (Theorem 2), this assumption is satisfied.

According to Theorem 4, we can now find a unique position of S^* that will ensure smoothness at S. ∎

In practice, we calculate this value by calculating the normals to Q_1, Q_2, and Q_3 at the point S, say $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$, respectively, and then solving the equation [5]

$$\|\mathbf{n}_1 \times \mathbf{n}_2\|^2 + \|\mathbf{n}_1 \times \mathbf{n}_3\|^2 + \|\mathbf{n}_2 \times \mathbf{n}_3\|^2 = 0. \tag{14}$$

This is an equation of degree 6 in t which, according to Theorem 5, will have a unique real solution.

SELECTION OF THE DIRECTION VECTOR

Once a nonproblematic direction vector is given, we can always determine the parameter t to ensure smoothness at the point S, according to Theorem 5. But how do we select a direction vector? To solve this problem, we first need to find a way to compare different approximations, i.e., we need to define the term "best approximation".

One approach could be to require a direction vector that may ensure smoothness at the vertices A, B, and C, but we do not have enough freedom. It is possible to obtain smoothness at one of these points, but numerical experiments show that the solution is not unique, and we loose symmetry if we choose just one of the vertices.

Method 1

The first method we shall consider is to require smoothness at S and to minimize a function such as $\max_i\{\sin^2\theta_i\}$ (or $\sum_{i=1}^{3}\sin^2\theta_i$), where θ_1 is the angle between the normal vectors to Q_2 and Q_3 at the point A, θ_2 is the angle between the normal vectors to Q_1 and Q_3 at the point B, and θ_3 is the angle between the normal vectors to Q_1 and Q_2 at the point C. The problem with this approach is that, once again, the solution is not unique, and it results in a very complex nonlinear optimization problem.

It is important to note that we are looking for any direction vector that gives satisfactory results. It is, therefore, not necessary to actually solve this optimization problem but, instead, we use a direct search method to find an acceptable direction, i.e., a direction that minimizes a discretization of the object function. We superimpose a grid on the upper half of the unit sphere with centre at S. The points on the sphere are given in spherical coordinates (Figure 2), with $\phi \in [0, \pi/2]$.

This interval can be subdivided into n smaller intervals of equal length. Similarly, $\theta \in [0, 2\pi]$, and this interval can also be subdivided into m smaller intervals. By selecting a fixed value for ϕ, we obtain a cone with its vertex at the point S, intersecting the sphere in a circle with radius $\sin\phi$. As ϕ increases, the radius of this circle increases, which means that the distance between the gridpoints on this circle increases as ϕ increases. To avoid this situation of unevenly spaced gridpoints, we select $m = \lceil 4n \sin\phi \rceil$ equispaced gridpoints on each of these circles. In this way, a set of direction vectors is defined, namely the vectors through S and the gridpoints. We then search through all these vectors to find the minimum value of $f(\phi, \theta)$, where $f(\phi, \theta) = \max\{\sin^2\theta_1, \sin^2\theta_2, \sin^2\theta_3\}$, and where each θ_i depends on the direction vector. It should also be mentioned that it is necessary to search through all the gridpoints, as the function f is not unimodal, and there is, therefore, no way to ensure that a more sophisticated method, such as dichotomous search [4], will converge to the global minimum of the function.

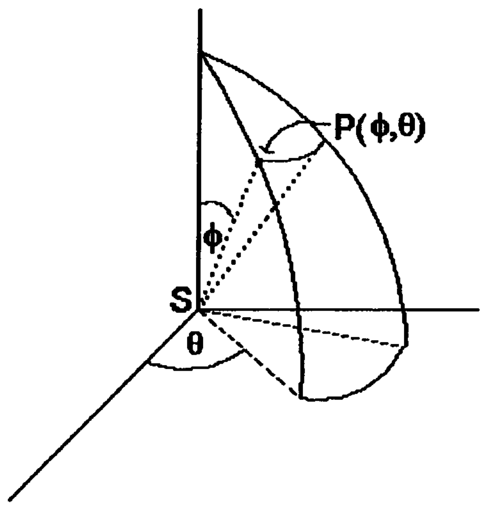

Figure 2. Spherical coordinates.

Method 2

Instead of using the requirement of smoothness at the point S to determine S^*, the point S^* can also be selected in the following way.

- Let α_1 be the angle between the normals to Q_2 and Q_3, α_2 the angle between the normals to Q_1 and Q_3, and α_3 the angle between the normals to Q_1 and Q_2 at the point S.
- Use a direct search method on the grid points of the sphere with radius t and centre S to find the value of S^* that minimizes the function

$$f(\phi, \theta) = \max_i \left\{\sin^2(\theta_i), \sin^2(\alpha_i)\right\}. \tag{15}$$

  Note that the angles θ_i and α_i now depend on three parameters (the direction as well as the parameter t).
- Change the radius of the sphere until a satisfactory solution is obtained. One possible starting value for the radius is the value of t, say t^*, obtained by the first method.

If the three given conics are quadric connected, and the quadric containing the three conics also contains the point S, the uniqueness of the parameter t will ensure that Method 1 results in the expected quadric for any nonproblematic direction. This, however, is not the case in Method 2, unless we use the distance between S and S^*, obtained in Method 1, as a starting value for the radius of the sphere in Method 2.

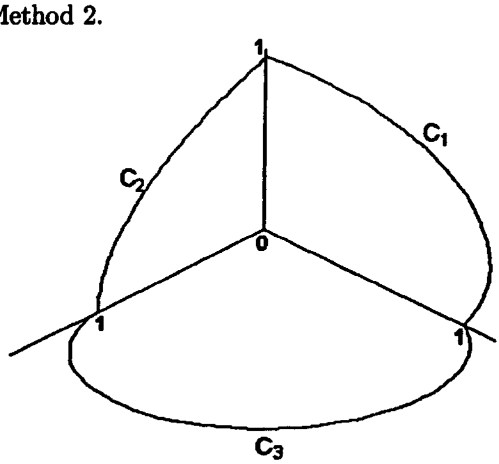

Figure 3. Example 1, given conics.

EXAMPLES

EXAMPLE 1. Suppose that the following three conics are given (see Figure 3):

$$C_1 : 2y^2 + 4z^2 - 2yz + y - z - 3 = 0,$$
$$C_2 : x^2 + 5z^2 + 4x - 5 = 0,$$
$$C_3 : 4x^2 + 6y^2 + 2xy - 7x - 9y + 3 = 0.$$

Application of Method 1 with $n = 20$ yields the following three quadric patches:

$$Q_1 : 2.141x^2 + 2y^2 + 4z^2 - 2.426xy - 0.055xz - 2yz - 0.659x + y - z - 3 = 0,$$
$$Q_2 : x^2 + 5.690y^2 + 5z^2 - 4.628xy - 4.052yz + 4x - 2.011y - 5 = 0,$$
$$Q_3 : 4x^2 + 6y^2 + 5.272z^2 + 2xy - 1.553xz - 4.272yz - 7x - 9y + 1.553z + 3 = 0,$$

and $S^* = (0.99999, 0.53187, -0.44076)$.

Application of Method 2, starting with a sphere of radius r, where r is the distance from S to S^* obtained in Method 1, and increasing and reducing the radius with increments of 0.1, we obtain the following three quadric patches:

$$Q_1 : 2.105x^2 + 2y^2 + 4z^2 - 2.995xy + 0.189xz - 2yz - 0.299x + y - z - 3 = 0,$$
$$Q_2 : x^2 + 6.313y^2 + 5z^2 - 5.957xy - 4.115yz + 4x - 1.242y - 5 = 0,$$
$$Q_3 : 4x^2 + 6y^2 + 4.941z^2 + 2xy - 1.260xz - 3.753yz - 7x - 9y + 1.071z + 3 = 0,$$

and $S^* = (1.03673, 0.53331, -0.44076)$.

In order to compare the results obtained by the two methods, we compare the maximum angle, denoted by $\theta_{\max}$, as well as the value Σ, where

$$\Sigma = \sum_i \sin^2(\theta_i) + \sum_j \sin^2(\alpha_j) \tag{16}$$

or each method. Although we used a different norm to do the optimization, practical experience shows that Σ is a good indicator to decide which method to use. The results are given in Table 1.

Table 1. Results of Example 1.

Method 1		Method 2	
$\theta_1 = 33.98°$	$\alpha_1 = 0°$	$\theta_1 = 31.71°$	$\alpha_1 = 17.00°$
$\theta_2 = 24.63°$	$\alpha_2 = 0°$	$\theta_2 = 23.17°$	$\alpha_2 = 12.12°$
$\theta_3 = 33.55°$	$\alpha_3 = 0°$	$\theta_3 = 28.28°$	$\alpha_3 = 15.60°$
$\theta_{\max} = 33.98°$		$\theta_{\max} = 31.71°$	
$\Sigma = 0.791497$		$\Sigma = 0.857438$	

The quadric patches resulting from Method 1 are shown in Figure 4.

It is clear that the angles between the normals at the points A, B, and C, obtained by Method 2, were slightly reduced, but the angles at the point S were considerably increased. In this case, Method 1 gave better results, which can also be seen by comparing the Σ-values. This is not true in general, as the following example shows.

EXAMPLE 2. Suppose that the following three conics are given (see Figure 5):

$$C_1 : y^2 + z^2 - 1 = 0,$$
$$C_2 : x^2 + z^2 - 1 = 0,$$
$$C_3 : 2x^2 + 3y^2 + 4xy - 3x - 4y + 1 = 0.$$

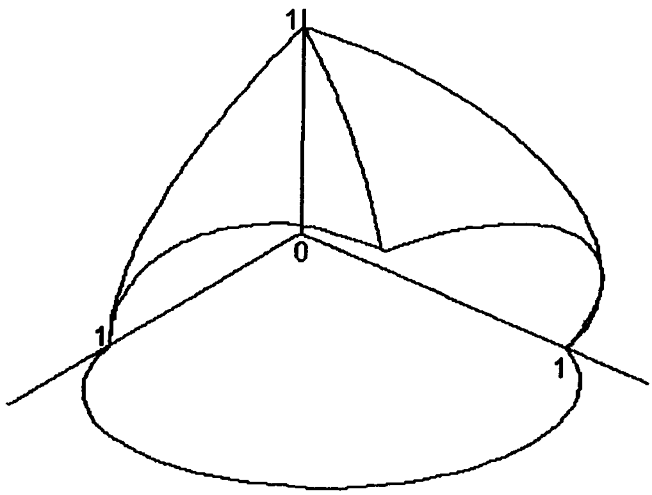

Figure 4. Example 1, quadric patches.

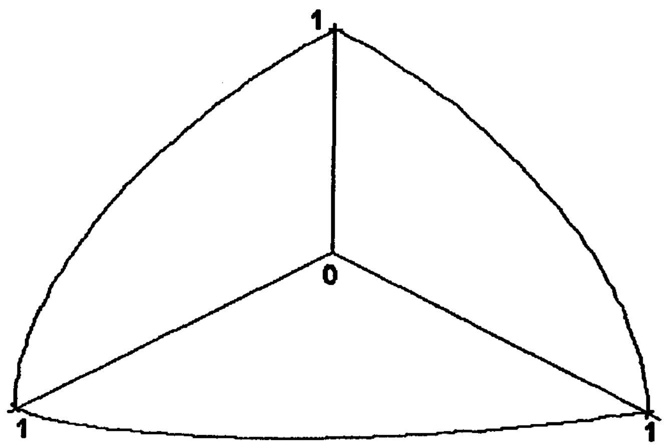

Figure 5. Example 2, given conics.

Applying Method 1, again with $n = 20$, we obtain

$$Q_1 : 0.681x^2 + y^2 + z^2 - 1.628xy - 2.375xz + 2.322x - 1 = 0,$$
$$Q_2 : x^2 + 1.685y^2 + z^2 - 2.586xy - 2.690yz + 2.591y - 1 = 0,$$
$$Q_3 : 2x^2 + 3y^2 + 2.653z^2 + 4xy - 5.295xz - 5.656yz - 3x - 4y + 5.297z + 1 = 0,$$

with $S^* = (1.15171, 0.70226, 1.02630)$.

Method 2, again with the appropriate starting value, results in

$$Q_1 : 0.596x^2 + y^2 + z^2 - 1.533xy - 2.141xz + 2.350x - 1 = 0,$$
$$Q_2 : x^2 + 2.023y^2 + z^2 - 2.884xy - 2.885yz + 2.746y - 1 = 0,$$
$$Q_3 : 2x^2 + 3y^2 + 2.621z^2 + 4xy - 5.401xz - 5.561yz - 3x - 4y + 5.341z + 1 = 0,$$

with $S^* = (1.19697, 0.61343, 1.03415)$.

A comparison between the two methods can be found in Table 2.

CONCLUSION

In this paper, the effect of various selections of the free parameters in the construction of Baart and McLeod were investigated. It was shown that, by ensuring smoothness at the point S, the resulting approximation is exact in the case where the given sculptured patch is itself quadric.

Table 2. Results of Example 2.

Method 1	Method 2
$\theta_1 = 0.20^\circ \quad \alpha_1 = 0^\circ$	$\theta_1 = 5.24^\circ \quad \alpha_1 = 0.62^\circ$
$\theta_2 = 11.74^\circ \quad \alpha_2 = 0^\circ$	$\theta_2 = 7.11^\circ \quad \alpha_2 = 1.86^\circ$
$\theta_3 = 3.20^\circ \quad \alpha_3 = 0^\circ$	$\theta_3 = 4.37^\circ \quad \alpha_3 = 1.49^\circ$
$\theta_{\max} = 11.74^\circ$	$\theta_{\max} = 7.11^\circ$
$\Sigma = 0.044515$	$\Sigma = 0.031303$

Some suitable norm for measuring the fit of the approximating quadric patches to the given surface is still necessary in order to select a "best" direction vector. Currently, we are investigating the minimization of the angles between the normals along the interior boundary conics, in order to get as close as possible to G^1 continuity along these curves [5,6].

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